

A remark on the partial regularity of a suitable weak solution to the Navier-Stokes Cauchy problem

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Abstract - Starting from the partial regularity results for suitable weak solutions to the Navier-Stokes Cauchy problem by Caffarelli, Kohn and Nirenberg [1], as a corollary, under suitable assumptions of local character on the initial data, we prove a behavior in time of the L_{loc}^∞ -norm of the solution in a neighborhood of $t = 0$. The behavior is the same as for the resolvent operator associated to the Stokes operator. Besides its own interest, the result is a main tool to study the spatial decay estimates of a suitable weak solution, performed in paper [2].

Keywords: Navier-Stokes equations, suitable weak solutions, partial regularity.

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1 Introduction

In this note we study the partial regularity of suitable weak solutions to the Navier-Stokes initial value problem:

$$\begin{aligned} v_t + v \cdot \nabla v + \nabla \pi_v &= \Delta v, \quad \nabla \cdot v = 0, \quad \text{in } (0, T) \times \mathbb{R}^3, \\ v(0, x) &= v_o(x) \quad \text{on } \{0\} \times \mathbb{R}^3. \end{aligned} \quad (1.1)$$

In system (1.1) v is the kinetic field, π_v is the pressure field, $v_t := \frac{\partial}{\partial t}v$ and $v \cdot \nabla v := v_k \frac{\partial}{\partial x_k}v$. In order to highlight the main ideas we assume zero body force, as well as we restrict ourselves to the Cauchy problem. However, the initial boundary value problem will be considered in a forthcoming paper.

The symbol $\mathcal{C}_0(\mathbb{R}^3)$ stands for the subset of $C_0^\infty(\mathbb{R}^3)$ whose elements are divergence free. We set $J^2(\mathbb{R}^3) := \text{completion of } \mathcal{C}_0(\mathbb{R}^3) \text{ with respect to the } L^2\text{-norm}$, and $J^{1,2}(\mathbb{R}^3) := \text{completion of } \mathcal{C}_0(\mathbb{R}^3) \text{ with respect to the } W^{1,2}(\mathbb{R}^3)\text{-norm}$.

We set $(u, g) := \int_{\mathbb{R}^3} u \cdot g dx$.

In the present note, assuming $v_o \in J^2(\mathbb{R}^3)$, and, for $x_0 \in \mathbb{R}^3$ and $R_0 > 0$,

$$\mathcal{E}_0(x_0, R_0) := \text{ess sup}_{B(x_0, R_0)} \|v_o\|_{w(x)} := \left\| \left(\int_{\mathbb{R}^3} \frac{|v_o(y)|^2}{|x - y|} dy \right)^{\frac{1}{2}} \right\|_{L^\infty(B(x_0, R_0))} \quad (1.2)$$

small in a suitable sense, where the smallness is independent of $B(x_0, R_0)$ and of v_o , we prove, in a neighborhood of $t = 0$, a time-weighted estimate of the L_{loc}^∞ -norm of a suitable weak solution (hence local regularity for all $t > 0$).

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Our result has to be put in the context of the general results obtained by Caffarelli-Kohn-Nirenberg in [1], that are crucial for our aims, although other results related to sufficient conditions for regularity could be employed (see e.g. [3], [4], [5], [12], [14]).

To better state our main result, we introduce some definitions and notation. This is done following as much as possible the ones in [1].

Definition 1.1 A pair (v, π_v) , such that $v : (0, T) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $\pi_v : (0, T) \times \mathbb{R}^3 \rightarrow \mathbb{R}$, is said a weak solution to problem (1.1) if

i) for all $T > 0$, $v \in L^2(0, T; J^{1,2}(\mathbb{R}^3))$ and $\pi_v \in L^{\frac{5}{3}}((0, T) \times \mathbb{R}^3)$,

$$\|v(t)\|_2^2 + 2 \int_s^t \|\nabla v(\tau)\|_2^2 d\tau \leq \|v(s)\|_2^2, \quad \text{for all } t \geq s, \text{ for } s = 0 \text{ and a.e. in } s \geq 0,$$

ii) $\lim_{t \rightarrow 0} \|v(t) - v_0\|_2 = 0$,

iii) for all $t, s \in (0, T)$, the pair (v, π_v) satisfies the equation:

$$\int_s^t \left[(v, \varphi_\tau) - (\nabla v, \nabla \varphi) + (v \cdot \nabla \varphi, v) + (\pi_v, \nabla \cdot \varphi) \right] d\tau + (v(s), \varphi(s)) = (v(t), \varphi(t)),$$

for all $\varphi \in C_0^1([0, T] \times \mathbb{R}^3)$.

Definition 1.2 A pair (v, π_v) is said a suitable weak solution if it is a weak solution in the sense of the Definition 1.1 and, moreover,

$$\begin{aligned} \int_{\mathbb{R}^3} |v(t)|^2 \phi(t) dx + 2 \int_s^t \int_{\mathbb{R}^3} |\nabla v(\tau)|^2 \phi dx d\tau &\leq \int_{\mathbb{R}^3} |v(s)|^2 \phi(s) dx \\ &+ \int_s^t \int_{\mathbb{R}^3} |v|^2 (\phi_\tau + \Delta \phi) dx d\tau + \int_s^t \int_{\mathbb{R}^3} (|v|^2 + 2\pi_v) v \cdot \nabla \phi dx d\tau, \end{aligned} \tag{1.3}$$

for all $t \geq s$, for $s = 0$ and a.e. in $s \geq 0$, and for all nonnegative $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$.

In [1] and [11] the following existence result is proved:

Theorem 1.1 For all $v_0 \in J^2(\mathbb{R}^3)$ there exists a suitable weak solution.

Concerning the regularity of a suitable weak solution, we begin by giving the following

Definition 1.3 We say that (t, x) is a singular point for a weak solution (v, π_v) if $v \notin L^\infty$ in any neighborhood of (t, x) ; the remaining points, where $v \in L^\infty(I(t, x))$ for some neighborhood $I(t, x)$, are called regular.

We introduce the parabolic cylinders

$$Q_r(t, x) := \{(\tau, y) : t - r^2 < \tau < t \text{ and } |y - x| < r\},$$

and

$$Q_r^*(t, x) := \{(\tau, y) : t - \frac{7}{8}r^2 < \tau < t + \frac{1}{8}r^2 \text{ and } |y - x| < r\},$$

that is $Q_r^*(t, x) = Q_r(t + \frac{1}{8}r^2, x)$. Moreover, for $r \in (0, t^{\frac{1}{2}})$, we set

$$M(r) := r^{-2} \iint_{Q_r} (|v|^3 + |v||\pi_v|) dy d\tau + r^{-\frac{13}{4}} \int_{t-r^2}^t \left(\int_{|x-y|<r} |\pi_v| dy \right)^{\frac{5}{4}} d\tau,$$

where, for simplicity, we suppress the dependence on (t, x) .

In [1], to achieve the regularity of a suitable weak solution two sufficient conditions are given. The first one is Proposition 1 (or Corollary 1) in [1]:

Proposition 1.1 *Let (v, π_v) be a suitable weak solution in some parabolic cylinder $Q_r(t, x)$. There exist $\varepsilon_1 > 0$ and $c_0 > 0$ independent of (v, π_v) such that, if $M(r) \leq \varepsilon_1$, then*

$$|v(\tau, y)| \leq c_1^{\frac{1}{2}} r^{-1}, \text{ a.e. in } (\tau, y) \in Q_{\frac{r}{2}}(t, x), \quad (1.4)$$

where $c_1 := c_0 \varepsilon_1^{\frac{2}{3}}$. In particular, a suitable weak solution v is regular in $Q_{\frac{r}{2}}(t, x)$.

Since in the statement of Proposition 1.1 it is required $M(r) \leq \varepsilon_1$ for all $r \in (0, t^{\frac{1}{2}})$, and since $M(r)$ is in integral form, we do not prejudice the problem giving the definition of $M(r)$ for $r \in (0, t^{\frac{1}{2}}]$.

The second sufficient condition is Proposition 2 in [1]:

Proposition 1.2 *There is a constant $\varepsilon_3 > 0$ with the following property. If (v, π_v) is a suitable weak solution in some parabolic cylinder $Q_r^*(t, x)$ and*

$$\limsup_{r \rightarrow 0} r^{-1} \iint_{Q_r^*} |\nabla v|^2 dy d\tau \leq \varepsilon_3,$$

then (t, x) is a regular point.

In [1], the above propositions are crucial to deduce the regularity for a suitable weak solution. In their applications, in particular Theorem C and Theorem D, and Corollary on p.820, the regularity results express on the geometry of regular points, but either a continuous dependence on the initial data and a behavior in a neighborhood of $t = 0$ do not seem an immediate consequence. Of course, the same difficulties arise from the results of [3], [4], [5], [12] and [14].

The following Theorem 1.2, which is the chief result of this note, is related to the pointwise continuous dependence of the null solution, in the framework of partial regularity for a suitable weak solution.

From now on we will assume that $v_o \in J^2(\mathbb{R}^3)$ and, in the light of Theorem 1.1, denote by (v, π_v) a corresponding suitable weak solution.

Theorem 1.2 *Let (v, π_v) be a suitable weak solution. Then there exist absolute constants ε_1 , C_1 and C_2 such that if*

$$C_1 \mathcal{E}_0(x_0, R_0) < 1 \text{ and } C_2(\mathcal{E}_0^3 + \mathcal{E}_0^{\frac{5}{2}}) \leq \varepsilon_1, \quad (1.5)$$

then

$$|v(t, x)| \leq c(\mathcal{E}_0^3 + \mathcal{E}_0^{\frac{5}{2}})^{\frac{1}{3}} t^{-\frac{1}{2}}, \quad (1.6)$$

provided that (t, x) is a Lebesgue point with $\|v_o\|_{w(x)} < \infty$ and $x \in B(x_0, R_0)$.

In the above statement ε_1 is the same as in Proposition 1.1.

The proof of the theorem is based on Proposition 1.1. This is also made in [1], but our approach to the proposition is different. Indeed, we prove a weighted energy relation (the norm is $\|v(t)\|_{w(x)}$) which holds for $t > 0$, provided that $(1.5)_1$ holds (see estimate (2.3) in Proposition 2.1).

Theorem 1.2 is of primary importance as a premise to another paper, [2], concerning the space-time decay of suitable weak solutions, provided that the same behavior is assumed on the initial data.

We end the introduction with few remarks.

Our partial regularity result states that under the smallness assumption (1.5) almost all $(t, x) \in (0, T) \times B(x_0, R_0)$ are points of regularity for a suitable weak solution v . In particular, if $B(x_0, R_0) \equiv \mathbb{R}^3$, then we get a new sufficient condition for the existence of a global smooth solution.

Condition (1.2) becomes a norm. We remark that for all $v_o \in J^2(\mathbb{R}^3)$, by virtue of the Hardy-Littlewood-Sobolev theorem, we have $\|v_o\|_{w(x)} < \infty$ almost everywhere in $x \in \mathbb{R}^3$. Hence, any data in L^2 inherently defines a functional that we can assume as norm. Condition (1.5) is just the smallness of the norm $\|v_o\|_{w(x)}\|_{L^\infty(B(x_0, R_0))}$. We can find several sufficient conditions on $v_o \in J^2(\mathbb{R}^3)$ such that assumption (1.5) is verified. For example, if we consider the assumption in [1] (Corollary on p. 820), that is $v_o \in W^{1,2}(\mathbb{R}^3 - B_R)$, then $\|v_o\|_{w(x)}$ is a continuous function of x that we can make small outside a suitable ball of radius $R' \geq R$.

The partial regularity has a local character in the sense that condition (1.5) can be not satisfied on $\mathbb{R}^3 - B(x_0, R_0)$, and the regularity is ensured a.e. in $B(x_0, R_0)$. Another feature that expresses the local character of the result is the following: the pointwise behavior of our solution v is given in a neighborhood of $(0, x_0)$. As far as we know, this property, which is as the one of the solutions to the Stokes problem, is new.

Of course estimate (1.6) also gives a pointwise asymptotic behavior of the solution for large t . However such a behavior is not optimal under the assumption $v_o \in J^2(\mathbb{R}^3)$. Indeed for a suitable weak solution the pointwise asymptotic behavior is of the kind $O(\|v_o\|_2 t^{-\frac{3}{4}})$, according to the fact that a weak solution becomes smooth for $t > T_0$, where T_0 is connected with the L^2 -norm of v_o , and the behavior is governed by the L^2 -norm of the initial data (see [7]).

We conclude by observing that a weaker result can be deduced for the initial boundary value problem. In this case the local regularity is far from the boundary. The result will be object of a next paper. We attack the problem by using the arguments developed in [6] and new estimates on the pressure field deduced in [10].

2 Partial regularity results

Firstly we recall some results fundamental for our aims.

Lemma 2.1 *Suppose that $|x|^\beta u \in L^2(\mathbb{R}^3)$ and $|x|^\alpha \nabla u \in L^2(\mathbb{R}^3)$. Also*

- i) $r \geq 2$, $\gamma + \frac{3}{r} > 0$, $\alpha + \frac{3}{2} > 0$, $\beta + \frac{3}{2} > 0$, and $a \in [\frac{1}{2}, 1]$,*

ii) $\gamma + \frac{3}{r} = a(\alpha + \frac{1}{2}) + (1-a)(\beta + \frac{3}{2})$ (dimensional balance),

iii) $a(\alpha - 1) + (1-a)\beta \leq \gamma \leq a\alpha + (1-a)\beta$.

Then, with a constant c independent of u , the following inequality holds:

$$\| |x|^\gamma u \|_r \leq c \| |x|^\alpha \nabla u \|_2^a \| |x|^\beta u \|_2^{1-a}. \quad (2.1)$$

Proof. See [1] Lemma 7.1. □

Lemma 2.2 Assume that \mathbb{K} is a singular bounded transformation from L^p into L^p , $p \in (1, \infty)$, of Calderón-Zigmund kind. Then, \mathbb{K} is also a bounded transformation from L^p into L^p with respect to the measure $(\mu + |x|)^\alpha dx$, $\mu \geq 0$, provided that $\alpha \in (-n, n(p-1))$.

Proof. See [13] Theorem 1. □

For the reader's convenience, here we restate Proposition 1.1, that is Corollary 2 in [1] in a slightly different form, more convenient for our aims:

Theorem 2.1 Let (v, π_v) be a suitable weak solution in some parabolic cylinder $Q_r(t, x)$. There exist $\varepsilon_1 > 0$ and $c_0 > 0$ independent of (v, π_v) such that, if $M(r) \leq \bar{\varepsilon}_1 \leq \varepsilon_1$, then

$$|v(\tau, y)| \leq c_1^{\frac{1}{2}} r^{-1}, \text{ a.e. in } (\tau, y) \in Q_{\frac{r}{2}}(t, x), \quad (2.2)$$

where $c_1 := c_0 \bar{\varepsilon}_1^{\frac{2}{3}}$.

Proof. For the regularity result see Corollary 1 on p.776 of [1], and for formula $c_1 = c_0 \bar{\varepsilon}_1^{\frac{2}{3}}$ see (4.4) on p.789 of [1]. □

Theorem 1.2 is proved employing Theorem 2.1. So that our task is reduced to prove that, under the assumption (1.5), for all suitable weak solutions (v, π_v) the following holds

$$M(r) \leq c(\mathcal{E}_0^3 + \mathcal{E}_0^{\frac{5}{2}}).$$

To this end we will prove

Proposition 2.1 Let be $C_1 \mathcal{E}_0 < 1$. Then there exists a set D and a suitable weak solution (v, π_v) such that $\text{meas}(B(x_0, R_0) - D) = 0$ and

$$\|v(t)\|_{w(x)}^2 + c(\mathcal{E}_0) \int_0^t \|\nabla u(\tau)\|_{w(x)}^2 d\tau < c \|v_0\|_{w(x)}^2, \quad (2.3)$$

for all $t > 0$ and $x \in D$,

where $c(\mathcal{E}_0) > 1$.

We postpone the proof of the proposition to the next section and now we deduce the above implication.

Lemma 2.3 Assume that (v, π_v) is a suitable weak solution. Then the pressure field admits the following representation formula

$$\pi_v(t, x) = -D_{x_i} D_{x_j} \int_{\mathbb{R}^3} \mathcal{E}(x - y) v^i(y) v^j(y) dy, \text{ a.e. in } (t, x) \in (0, \infty) \times \mathbb{R}^3. \quad (2.4)$$

Proof. Since $\pi_v \in L^{\frac{5}{3}}((0, T) \times \mathbb{R}^3)$ and $v \in L^2(0, T; J^{1,2}(\mathbb{R}^3))$, there exists a zero Lebesgue measure set N such that, for all $t \in (0, T) - N$, $\|\pi_v(t)\|_{\frac{5}{3}} + \|v(t)\|_{1,2} < \infty$. To prove (2.4) we start from the formulation of weak solution to the Navier-Stokes Cauchy problem evaluated for $t \in (0, T) - N$ and for all $\varphi \in C_0^\infty([0, T) \times \mathbb{R}^3)$:

$$(v(t+\delta), \varphi(t+\delta)) - (v(t), \varphi(t)) = \int_t^{t+\delta} [(v, \varphi_\tau + \Delta\varphi) + (v \cdot \nabla\varphi, v) + (\pi_v, \nabla \cdot \varphi)] d\tau.$$

In particular setting $\varphi(\tau, x) := h(\tau)\nabla\psi(x)$, where $h(\tau) := \begin{cases} 1 & \text{for } \tau \leq t+\delta, \\ 0 & \text{for } \tau \geq t+2\delta, \end{cases}$ is a smooth cut-off function and $\psi(x) \in C_0^\infty(\mathbb{R}^3)$, since v is divergence free, we get

$$\int_t^{t+\delta} [(v \cdot \nabla\nabla\psi, v) + (\pi_v, \Delta\psi)] d\tau = 0.$$

Multiplying by $\frac{1}{\delta}$, in the limit as $\delta \rightarrow 0$, we deduce

$$(\pi_v, \Delta\psi) = -(v \otimes v, \nabla\nabla\psi), \text{ a.e. in } t > 0, \text{ for all } \psi \in C_0^\infty(\mathbb{R}^3). \quad (2.5)$$

A solution of equation (2.5) is given by

$$\bar{\pi} = -D_{x_i}D_{x_j} \int_{\mathbb{R}^3} \mathcal{E}(x-y)v^i(t,y)v^j(t,y)dy.$$

Since, for $t \in (0, T) - N$, we have $v \in J^{1,2}(\mathbb{R}^3) \subset L^{\frac{10}{3}}(\mathbb{R}^3)$, by virtue of Calderón-Zigmund theorem we get $\bar{\pi} \in L^{\frac{5}{3}}(\mathbb{R}^3)$. Hence, we deduce, almost everywhere in $t > 0$,

$$(\pi_v - \bar{\pi}, \Delta\psi) = 0, \text{ for all } \psi \in C_0^\infty(\mathbb{R}^3),$$

with $\pi_v - \bar{\pi} \in L^{\frac{5}{3}}(\mathbb{R}^3)$, so that $\pi_v \equiv \bar{\pi}$ and (2.4) is achieved. \square

If (2.3) holds, then, with the aid of the previous lemma, we are able to prove the following

Lemma 2.4 *Assume that (2.3) holds for a suitable weak solution (v, π_v) . Then $M(r) \leq C_2(\mathcal{E}_0^3 + \mathcal{E}_0^{\frac{5}{2}})$, for $r^2 \in (0, t)$.*

Proof. By virtue of our assumption (2.3), by virtue of representation formula (2.4) and Lemma 2.2, a.e. in $t > 0$, we get that

$$\|\pi_v(t)|x-y|^{-\frac{4}{3}}\|_{\frac{3}{2}} \leq c\|v(t)\|_3\|x-y|^{-\frac{2}{3}}\|_3^2. \quad (2.6)$$

Applying Hölder's inequality, from (2.6) and from Lemma 2.1 we get

$$\begin{aligned} r^{-2} \int_{t-r^2}^t \int_{|x-y|<r} [|v|^3 + |v||\pi_v|] dy d\tau &\leq c \int_{t-r^2}^t \left[\left\| \frac{v(\tau)}{|x-y|^{\frac{2}{3}}} \right\|_3^3 + \left\| \frac{v(\tau)}{|x-y|^{\frac{2}{3}}} \right\|_3 \left\| \frac{\pi_v(\tau)}{|x-y|^{\frac{2}{3}}} \right\|_{\frac{3}{2}} \right] d\tau \\ &\leq c \int_{t-r^2}^t \left\| \frac{v(\tau)}{|x-y|^{\frac{1}{2}}} \right\|_2^2 \left\| \frac{\nabla v(\tau)}{|x-y|^{\frac{1}{2}}} \right\|_2^2 d\tau \leq c\|v\|_{w(x)}^3, \end{aligned} \quad (2.7)$$

for all $t > 0$ and $t - r^2 > 0$.

Considering the last term on the right-hand side of $M(r)$, applying twice Hölder's inequality, (2.6), we get

$$\begin{aligned}
r^{-\frac{13}{4}} \int_{t-r^2}^t \left[\int_{|x-y|<r} |\pi_v(\tau, y)| dy \right]^{\frac{5}{4}} d\tau &\leq cr^{-\frac{1}{3}} \int_{t-r^2}^t \left[\left\| \frac{\pi_v(\tau)}{|x-y|^{\frac{4}{3}}} \right\|_{\frac{3}{2}} \right]^{\frac{5}{4}} d\tau \\
&\leq cr^{-\frac{1}{3}} \int_{t-r^2}^t \left\| \frac{v(\tau)}{|x-y|^{\frac{1}{2}}} \right\|_{\frac{5}{2}}^{\frac{5}{6}} \left\| \frac{\nabla v(\tau)}{|x-y|^{\frac{1}{2}}} \right\|_{\frac{5}{3}}^{\frac{5}{3}} d\tau \\
&\leq c \|v_o\|_{w(x)}^{\frac{5}{2}}, \quad \text{for all } t > 0 \text{ and } t - r^2 > 0.
\end{aligned} \tag{2.8}$$

Hence (2.7) and (2.8) imply, for a suitable constant C_2 , that

$$M(r) \leq C_2(\mathcal{E}_0^3 + \mathcal{E}_0^{\frac{5}{2}}).$$

The lemma is proved. \square

3 Proof of Proposition 2.1

The aim of this section is to prove Proposition 2.1. We start with following result:

Lemma 3.1 *For all suitable weak solutions (v, π_v) , for $s = 0$ and almost everywhere in $s \geq 0$, we get*

$$\lim_{t \rightarrow s^+} \int_{\mathbb{R}^3} |v(t, x) - v(s, x)|^2 \phi dx = 0, \text{ for all } \phi \in L^\infty(\mathbb{R}^3). \tag{3.1}$$

Proof. It is well known that any weak solution, which satisfies the energy relation in the form i) of Definition 1.1, is right-continuous with values in $L^2(\mathbb{R}^3)$, for $s = 0$ and almost everywhere in $s \geq 0$. Hence, the limit property easily follows. \square

Lemma 3.2 *Under the assumption of Proposition 2.1, a suitable weak solution (v, π_v) satisfies the following weighted energy inequality:*

$$\int_{\mathbb{R}^3} \frac{|v(t, y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy + c(\mathcal{E}_0) \int_0^t \int_{\mathbb{R}^3} \frac{|\nabla v(\tau, y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy d\tau \leq \int_{\mathbb{R}^3} \frac{|v_o(y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}}, \tag{3.2}$$

for all $t > 0$, $\mu > 0$, and a.e. in $x \in B(x_0, R_0)$.

Proof. By virtue of our assumptions, almost everywhere in $x \in B(x_0, R_0)$, we have

$$\int_{\mathbb{R}^3} \frac{|v_o(y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} < c\mathcal{E}_0^2. \tag{3.3}$$

We define $\phi(\tau, y) := (|x-y|^2 + \mu^2)^{-\frac{1}{2}} h_R(y) k(\tau) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$, with h_R and k such that

$$h_R(y) := \begin{cases} 1 & \text{if } |y| \leq R \\ \in (0, 1) & \text{if } |y| \in [R, 2R] \\ 0 & \text{for } |y| \geq 2R \end{cases} \quad \text{and } k(\tau) := \begin{cases} 1 & \text{if } |\tau| \leq t \\ \in (0, 1) & \text{if } |\tau| \in [t, 2t] \\ 0 & \text{for } |\tau| \geq 2t \end{cases}.$$

Substituting this ϕ in (1.3), we get

$$\begin{aligned}
& \int_{\mathbb{R}^3} \frac{|v(t, y)|^2 h(y)}{(|x - y|^2 + \mu^2)^{\frac{1}{2}}} dy + 2 \int_0^t \int_{\mathbb{R}^3} \frac{|\nabla v(\tau, y)|^2 h(y)}{(|x - y|^2 + \mu^2)^{\frac{1}{2}}} dy d\tau + 3\mu^2 \int_0^t \int_{\mathbb{R}^3} \frac{|v(\tau, y)|^2 h(y)}{(|x - y|^2 + \mu^2)^{\frac{3}{2}}} dy d\tau \\
& \leq \int_{\mathbb{R}^3} \frac{|v_\circ(y)|^2 h(y)}{(|x - y|^2 + \mu^2)^{\frac{1}{2}}} dy + \int_0^t \int_{\mathbb{R}^3} \frac{|v(\tau, y)|^2 h(y) v(y) \cdot (x - y)}{(|x - y|^2 + \mu^2)^{\frac{3}{2}}} dy d\tau \\
& \quad + \int_0^t \int_{\mathbb{R}^3} \frac{\pi_v(y) h(y) v(y) \cdot (x - y)}{(|x - y|^2 + \mu^2)^{\frac{3}{2}}} dy d\tau + o(R) \\
& := \int_{\mathbb{R}^3} \frac{|v_\circ(y)|^2 h(y)}{(|x - y|^2 + \mu^2)^{\frac{1}{2}}} dy + I_1(t, x) + I_2(t, x) + o(R),
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
o(R) &:= \int_0^t \int_{\mathbb{R}^3} |v|^2 \left[2 \nabla h_R \cdot \nabla (|x - y|^2 + \mu^2)^{-\frac{1}{2}} + \frac{\Delta h_R}{(|x - y|^2 + \mu^2)^{\frac{1}{2}}} + \frac{v \cdot \nabla h_R}{(|x - y|^2 + \mu^2)^{\frac{1}{2}}} \right] dy d\tau \\
& \quad + \int_0^t \int_{\mathbb{R}^3} \frac{\pi_v v \cdot \nabla h_R}{(|x - y|^2 + \mu^2)^{\frac{1}{2}}} dy d\tau.
\end{aligned}$$

We estimate the terms I_i , $i = 1, 2$. Since $\mu > 0$, by virtue of the integrability properties of a suitable weak solution, applying Hölder's inequality and Lemma 2.1 we get

$$|I_1(t, x)| \leq \left\| \frac{v}{(|x - y|^2 + \mu^2)^{\frac{1}{4}}} \right\|_3^3 \leq c \left\| \frac{v}{(|x - y|^2 + \mu^2)^{\frac{1}{4}}} \right\|_2 \left\| \frac{\nabla v}{(|x - y|^2 + \mu^2)^{\frac{1}{4}}} \right\|_2^2.$$

For I_2 applying the Hölder's inequality and Lemma 2.2, we obtain

$$|I_2(t, x)| \leq c \left\| \frac{v}{(|x - y|^2 + \mu^2)^{\frac{1}{4}}} \right\|_3^3 \left\| \frac{\pi_v}{(|x - y|^2 + \mu^2)^{\frac{1}{2}}} \right\|_{\frac{3}{2}}^{\frac{3}{2}} \leq c \left\| \frac{v}{(|x - y|^2 + \mu^2)^{\frac{1}{4}}} \right\|_3^3.$$

Hence, as in the previous case, applying Lemma 2.1, we get

$$|I_2(t, x)| \leq c \left\| \frac{v}{(|x - y|^2 + \mu^2)^{\frac{1}{4}}} \right\|_2 \left\| \frac{\nabla v}{(|x - y|^2 + \mu^2)^{\frac{1}{4}}} \right\|_2^2.$$

We increase the right hand side of (3.3) by employing the estimates obtained for I_i , $i = 1, 2$. Hence, by applying the Lebesgue dominate convergence theorem, in the limit as $R \rightarrow \infty$, for all $t > 0$ we deduce the inequality

$$\begin{aligned}
& \int_{\mathbb{R}^3} \frac{|v(t, y)|^2}{(|x - y|^2 + \mu^2)^{\frac{1}{2}}} dy + 2 \int_0^t \int_{\mathbb{R}^3} \frac{|\nabla v(\tau, y)|^2}{(|x - y|^2 + \mu^2)^{\frac{1}{2}}} dy d\tau \leq \int_{\mathbb{R}^3} \frac{|v_\circ(y)|^2}{(|x - y|^2 + \mu^2)^{\frac{1}{2}}} dy \\
& \quad + c \int_0^t \left\| \frac{v(\tau)}{(|x - y|^2 + \mu^2)^{\frac{1}{4}}} \right\|_2^2 \left\| \frac{\nabla v(\tau)}{(|x - y|^2 + \mu^2)^{\frac{1}{4}}} \right\|_2^2 d\tau.
\end{aligned} \tag{3.5}$$

Since, by virtue of Lemma 3.1, for all $\mu > 0$ $\frac{v(t,y)}{(|x-y|^2+\mu^2)^{\frac{1}{2}}}$ is right-continuous with values in $L^2(\mathbb{R}^3)$ for $s = 0$ and almost everywhere in $s \geq 0$, then, a.e. in $x \in B(x_0, R_0)$, there exists a $[0, \delta(\mu, x))$ in which

$$\int_{\mathbb{R}^3} \frac{|v(t, y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy \leq c\mathcal{E}_0^2, \quad t \in [0, \delta). \quad (3.6)$$

Hence, on $[0, \delta)$ estimate (3.5) becomes

$$\int_{\mathbb{R}^3} \frac{|v(t, y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy + (2 - c\mathcal{E}_0) \int_0^t \int_{\mathbb{R}^3} \frac{|\nabla v(\tau, y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy d\tau \leq \int_{\mathbb{R}^3} \frac{|v_\circ(y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy. \quad (3.7)$$

Estimate (3.7) implies

$$\int_{\mathbb{R}^3} \frac{|v(t, y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy \leq \int_{\mathbb{R}^3} \frac{|v_\circ(y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy < c\mathcal{E}_0^2.$$

Let us prove that this last estimate holds for all $t > 0$. Firstly, let us explicitly note that, for all $\mu > 0$, the function

$$f(t) = c \int_0^t \left\| \frac{v(\tau)}{(|x-y|^2 + \mu^2)^{\frac{1}{4}}} \right\|_2 \left\| \frac{\nabla v(\tau)}{(|x-y|^2 + \mu^2)^{\frac{1}{4}}} \right\|_2^2 d\tau$$

is uniformly continuous. Hence there exists $\eta > 0$ such that

$$|t_1 - t_2| < \eta \Rightarrow |f(t_1) - f(t_2)| < c\mathcal{E}_0^2 - \int_{\mathbb{R}^3} \frac{|v_\circ(y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy.$$

We claim that estimate (3.6) holds in $[\delta, \delta + \eta)$. Assuming the contrary, there exists $\bar{t} \in [\delta, \delta + \eta)$ such that

$$\int_{\mathbb{R}^3} \frac{|v(\bar{t}, y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy > c\mathcal{E}_0^2. \quad (3.8)$$

On the other hand, the validity of (3.5) yields

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{|v(\bar{t}, y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy + 2 \int_0^{\bar{t}} \int_{\mathbb{R}^3} \frac{|\nabla v(\tau, y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy d\tau &\leq \int_{\mathbb{R}^3} \frac{|v_\circ(y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy \\ &+ (f(\bar{t}) - f(\delta)) + f(\delta). \end{aligned}$$

Since via (3.6) we get

$$f(\delta) \leq c\mathcal{E}_0 \int_0^\delta \int_{\mathbb{R}^3} \frac{|\nabla v(\tau, y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy d\tau < c\mathcal{E}_0 \int_0^{\bar{t}} \int_{\mathbb{R}^3} \frac{|\nabla v(\tau, y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy d\tau,$$

employing the uniform continuity condition, we find

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{|v(\bar{t}, y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy + (2 - c\mathcal{E}_0) \int_0^{\bar{t}} \int_{\mathbb{R}^3} \frac{|\nabla v(\tau, y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy d\tau \\ \leq \int_{\mathbb{R}^3} \frac{|v_0(y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy + f(\bar{t}) - f(\delta) < c\mathcal{E}_0^2, \end{aligned}$$

which contradicts (3.8). Since the arguments are independent of δ , we have proved estimate (3.2). \square

Corollary 3.1 *Under the assumption of Proposition 2.1, almost everywhere in $x \in B(x_0, R_0)$, we get*

$$\int_{\mathbb{R}^3} \frac{|v(t, y)|^2}{|x-y|} dy + c(\mathcal{E}_0) \int_0^t \int_{\mathbb{R}^3} \frac{|\nabla v(\tau, y)|^2}{|x-y|} dy d\tau \leq c \int_{\mathbb{R}^3} \frac{|v_0|^2}{|x-y|} dy, \text{ for all } t > 0. \quad (3.9)$$

Proof. The thesis is an easy consequence of the following remark: the families of functions

$$\left\{ \int_0^t \int_{\mathbb{R}^3} \frac{|\nabla v(\tau, y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy d\tau \right\} \text{ and } \left\{ \int_{\mathbb{R}^3} \frac{|v(t, y)|^2}{(|x-y|^2 + \mu^2)^{\frac{1}{2}}} dy \right\}$$

are monotone in $\mu > 0$. Hence, by virtue of the Beppo Levi's theorem, in the limit as $\mu \rightarrow 0$, we deduce (3.9). \square

4 Proof of Theorem 1.2

We consider a Lebesgue point (t, x) with $x \in D$ such that (2.3) holds. Let us consider the parabolic cylinder $Q_{\sqrt{t}}(\frac{7}{6}t, x)$. Since our assumptions on v_0 and x ensure that Proposition 2.1 holds, by virtue of Lemma 2.4 and Theorem 2.1 we get

$$|v(\tau, y)| \leq 2(c_1)^{\frac{1}{2}} t^{-\frac{1}{2}}, \text{ in } Q_{\frac{\sqrt{t}}{2}}(\frac{7}{6}t, x),$$

provided that (τ, y) is Lebesgue point. Since we are referring to (t, x) which belongs to $Q_{\frac{\sqrt{t}}{2}}(\frac{7}{6}t, x)$ and recalling the expression of $c_1 = c_0(\mathcal{E}_0^3 + \mathcal{E}_0^{\frac{5}{2}})^{\frac{2}{3}}$, we have proved the theorem. \square

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